Persistent spin currents induced by a spatially-dependent magnetic field in a spin-1/2 Heisenberg antiferromagnetic ring

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Abstract

We show that a spatially-dependent magnetic field can induce a persistent spin current in a spin-1/2 Heisenberg antiferromagnetic ring, proportional to the solid angle subtended by the magnetic field on a unit sphere. The result is a direct consequence of Berry "parallel transport" in space. The magnitude of the spin current is determined by the ratio of longitudinal and transverse exchange interactions $J_{\parallel}/J_{\perp}$ and by the magnetic field. For large magnetic fields the Zeeman energy strongly renormalizes the Ising term giving rise to a maximum spin current. In the limit of $J_{\parallel}/J_{\perp} \ll 1$ the amplitude of the current behaves like $1-(J_{\parallel}/J_{\perp})^2$. In the opposite limit $\pi J_{\perp} > J_{\parallel} > J_{\perp}$ the amplitude scales as $\sqrt{J_{\parallel}/J_{\perp}}\exp(-\sqrt{J_{\parallel}/J_{\perp}})$.

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The generation of persistent currents in a ring is a consequence of the sensitivity of a wavefunction to its boundary conditions. The basic idea of persistent currents was put forward a few decades ago [1–3]. The effect of a magnetic flux threading a metallic ring, such that the magnetic field at the ring vanishes, can be replaced by a twist of the wavefunction boundary conditions. The change in these boundary conditions gives rise to an asymmetry between the number of electrons with $K > K_F$ and $K < -K_F$, where K_F denotes Fermi momentum. As a result, an electric current is induced in mesoscopic ring structures. For large ring structures, in the thermodynamic limit, the current vanishes; in agreement with the fact that no spontaneous broken chirality occurs in non-interacting systems. A new interest in this problem has arisen for non-Fermi liquids. In particular, for Luttinger liquids novel behavior has recently been proposed [4–7].

In the last ten years the "parallel transport" idea introduced by Berry [8] for adiabatic processes has provided new insight into currents induced by topological effects. A typical situation considered by Berry was a spin-1/2 electron in a time periodic magnetic field. Assuming adiabatic motion Berry showed that when an electron returns to its initial position the wavefunction accumulates a nontrivial phase equal to the angle of rotation of the spin-1/2 angular momentum on a sphere. This is the Berry phase [8–10].

Recently it has been shown [11] that for an integer spin $S = 1$ ferromagnet the Berry phase combined with the spin-orbit effect can give rise to a spin current and a corresponding electric field. Here we propose a new way of generating a spin current, when a spatially periodic magnetic field acts on a *spin-1/2 antiferromagnetic ring a Berry phase in space is induced* giving rise to a twist on the wavefunction boundary condition. As a consequence a persistent spin current is induced. This result can be understood within the Berry parallel transport idea since a spin-1/2 antiferromagnet is Lorentz invariant.

We consider a spin-1/2 antiferromagnetic Heisenberg ring of circumference L in the presence of a spatially-dependent magnetic field of the "crown shape" form, $\vec{h}(x) = h_0(\sin\theta(x)\cos\phi(x), \sin\theta(x)\sin\phi(x), \cos\theta(x))$, where $\theta(x)$ and $\phi(x)$ are polar $-\pi \le \phi(x) \le \pi$, and azimuthal $0 < \theta(x) \le \theta_0 < \pi/2$ angles such that $\phi(x) = \phi(x + L)$, see Fig. 1. We

restrict $\theta(x)$ to the "northern" hemisphere which obeys $\theta(x) = \theta(x+L)$. The Hamiltonian for the spin-1/2 antiferromagnet in an external magnetic field is:

$$H = \frac{\tilde{J}_\perp}{2}\sum_x (S_+(x)S_-(x+1)+S_-(x+1)S_+(x))+\tilde{J}_\parallel \sum_x S_3(x)S_3(x+1)+g\mu_B \sum_x \vec{h}(x)\cdot\vec{S}(x), \quad (1)$$

where $\tilde{J}_\perp$ and $\tilde{J}_\parallel$ are the transverse and longitudinal exchange coupling constants.

We rotate $\vec{S}(x)$ in the direction of the magnetic field $\vec{h}(x)$ using the following unitary transformation:

$$\hat{U} = \Pi_{x=0}^{N-1} \exp\left(-i\frac{\phi(x)}{2}\sigma_3\right)\exp\left(-i\frac{\theta(x)}{2}\sigma_2\right), \quad (2)$$

where the circumference of the ring is $L = Na$ and σ_2 and σ_3 are Pauli spin-1/2 matrices. As a result of the unitary transformation Eq. (2), the Hamiltonian H is transformed to $\tilde{H} = \hat{U}^\dagger H \hat{U}$. Due to the periodicity of the angles $\theta(x)$ and $\phi(x)$ the transformed wavefunction obeys the same boundary condition as the original wavefunction.

The transformed Hamiltonian $\tilde{H}$ takes the form,

$$\tilde{H} \simeq \frac{J_\perp}{2}\sum_{x=0}^{N-1}\Big(S_+(x)e^{iA(x,x+1)}S_-(x+1)+S_-(x)e^{-iA(x,x+1)}S_+(x+1)\Big)$$
$$+J_\parallel \sum_{x=0}^{N-1} S_3(x)S_3(x+1) + g\mu_B h_0 \sum_x S_3(x). \quad (3)$$

In Eq. (3) $A(x,x+1)$ is the Berry connection:

$$A(x,x+1) \equiv A_\phi d\phi = \frac{1}{2}(1+\cos\theta(x))\frac{d\phi}{dx}dx. \quad (4)$$

To obtain Eq. (4) we have restricted $\theta(x)$ to the range $0 < \theta(x) < \theta_0 \leq \pi/4$, where θ_0 denotes the maximum tilt angle of the "crown-shaped" magnetic field (Fig. 1). The Berry connection results from rotating the coordinate of the spin-1/2 operator. Moving around the ring, a phase proportional to the subtended solid angle accumulates.

Due to the unitary transformation the exchange couplings $\tilde{J}_\perp$, $\tilde{J}_\parallel$ are replaced by $J_\perp$ and $J_\parallel$, where $J_\parallel \equiv \tilde{J}_\parallel\langle\cos^2\theta(x)\rangle$, $J_\perp \equiv \tilde{J}_\perp((1+\langle\cos^2\theta(x)\rangle)/2)$ with $\langle\cos^2\theta(x)\rangle \equiv (1/\theta_0)\int_0^{\theta_0}\cos^2\theta d\theta = (1/2)(1+(\sin 2\theta_0/2\theta_0))$.

Following Ref. [12] we use a Jordan-Wigner transformation to replace the spin-1/2 $\vec{S}(x)$ operator by the spinless fermions $c^\dagger$, $c(x)$:

$$S_-(x) = e^{-i\pi \sum_{x=0}^{N-1} n(x)} c(x), \quad S_+(x) = [S_-(x)]^\dagger, \quad S_3(x) = c^\dagger(x)c(x) - 1/2, \quad n(x) = c^\dagger(x)c(x). \tag{5}$$

It is convenient to replace the fermion $c(x)$ by $d(x)$, $c(x) = e^{i\Gamma(x)}d(x)$, $c^\dagger(x) = d^\dagger(x)e^{-i\Gamma(x)}$. In addition, we choose $A(x, x+1) + \pi(n(x) - 1) = \Gamma(x) - \Gamma(x+1)$. As a result the Hamiltonian in Eq. (3) is replaced by:

$$\tilde{H} = -\frac{J_\perp}{2} \sum_{x=0}^{N-1} (d_+(x)d(x+1) + H.c.) + J_\parallel \sum_{x=0}^{N-1} \left(n(x) - \frac{1}{2}\right)\left(n(x+1) - \frac{1}{2}\right)$$
$$+ g\mu_B h_0 \sum_x \left(n(x) - \frac{1}{2}\right). \tag{6}$$

To obtain the form given in Eq. (6), we require that the phase $\Gamma(x)$ obeys the condition

$$e^{i[A(x,x+1) + \pi(n(x)-1) - \Gamma(x) + \Gamma(x+1)]} = 1. \tag{7}$$

The periodicity of the spin operator $\vec{S}(x) = \vec{S}(x+L)$ together with the condition on the Berry connection gives rise to a *twisted boundary condition* for the $d(x)$, $d^\dagger(x)$ fermions:

$$d(x+N) = -e^{i\sum_{J=0}^{N-1} A(x+J, x+J+1)} e^{i2\pi N_F} e^{-i\pi(N+1)} d(x), \tag{8a}$$

where $N_F = \sum_{x=0}^{N-1} n(x)$ and $L = Na$. Replacing the sum in Eq. (8a) by a continuous integral we find, using Stokes theorem, that

$$\sum_{J=0}^{N-1} A(x+J, x+1+J) \simeq \oint A_\phi d\phi = \int_0^{\theta_0} \int_{-\pi}^{\pi} \left(-\frac{\partial A_\phi}{\partial \theta}\right) d\phi d\theta = \pi(1 - \cos\theta_0) \equiv \Omega(\theta_0). \tag{8b}$$

Here, $\Omega(\theta_0)$ measures the twist on the boundary conditions. Consequently, the equation for $d(x)$ is replaced by

$$d(x+L) = -e^{i\pi(\Omega(\theta_0)/\pi - 1)} d(x) e^{i\pi N} = -e^{i\pi(\delta - 1)} d(x), \tag{8c}$$

where $\delta = \Omega(\theta_0)/\pi + \Delta_N$, $\Delta_N = 0$ if N is even and $\Delta_N = 1$ if N is odd. Equation (8c) expresses the twist of the boundary condition. Consequently, the "twist" of the boundary

condition is equal to the solid angle $\Omega(\theta_0)$, see Fig. 1. Equations (8b) and (8c) represent one of our major results: a geometrical rotation of the spin-1/2 coordinate gives rise to a Berry phase which changes the boundary conditions for the wavefunction.

We construct the continuum representation of the Hamiltonian $\tilde{H}$. The fermion operator $d(x)$ is replaced by $\psi(x)$:

$$d(x) = \sqrt{a}\psi(x) = \sqrt{a}[e^{ik_F^0 x}e^{-i(2\pi/L)(\delta-1)x}R(x) + e^{-ik_F^0 x}e^{-i(2\pi/L)(\delta-1)x}L(x), \tag{9}$$

where $k_F^0 = \pi/2a$ and $R(x)$ and $L(x)$ are the right and left moving operators at half filling:

$$R(x) = \frac{1}{\sqrt{2\pi a}}e^{i\alpha_R}e^{i(2\pi/L)(\hat{N}_R-1/2)x}e^{i\sqrt{4\pi}\hat{\theta}_R},$$

$$L(x) = \frac{1}{\sqrt{2\pi a}}e^{-i\alpha_L}e^{-i(2\pi/L)(\hat{N}_L-1/2)x}e^{-i\sqrt{4\pi}\hat{\theta}_L}. \tag{10}$$

The representation given in Eq. (10) has been introduced in Refs. [7,13]. $\hat{\theta}_R$ and $\hat{\theta}_L$ are the particle hole excitations (non-zero modes), $\hat{N}_R$ and $\hat{N}_L$ are the *zero mode* charge operators. $\hat{N}_R$ and $\hat{N}_L$ measure the additional charge with respect to the half-filled spinless Fermi sea. α_R and α_L represent the zero mode coordinates which are canonical conjugates to $\hat{N}_R$ and $\hat{N}_L$, respectively:

$$[\alpha_R, \hat{N}_R] = [-\alpha_L, \hat{N}_L] = i, \quad [\alpha, \hat{J}] = 2i, \quad [\beta, \hat{Q}] = 2i, \quad \hat{J} = \hat{N}_L - \hat{N}_R, \quad \hat{Q} = \hat{N}_L + \hat{N}_R,$$

$$\alpha = \alpha_R + \alpha_L, \quad \beta = \alpha_L - \alpha_R. \tag{11}$$

The complete set of zero mode eigenfunctions are given by

$$\langle\alpha_R|N_R\rangle = \frac{1}{\sqrt{2\pi}}e^{i\alpha_R N_R}, \quad \langle\alpha_L|N_L\rangle = \frac{1}{\sqrt{2\pi}}e^{i\alpha_L N_L} \tag{12a}$$

or

$$\langle\alpha|J\rangle = \frac{1}{\sqrt{4\pi}}e^{i\alpha J/2}, \quad \langle\beta|Q\rangle = \frac{1}{\sqrt{4\pi}}e^{i\beta Q/2}, \tag{12b}$$

where $\hat{N}_R|N_R\rangle = N_R|N_R\rangle$, $\hat{N}_L|N_L\rangle = N_L N_L\rangle$, $\hat{J}|J\rangle = J|J\rangle$ and $\hat{Q}|Q\rangle = Q|Q\rangle$.

We perform a unitary transformation which removes the chiral phase $(2\pi/L)(\delta-1)x$ from Eq. (9). The Fermi momentum $k_F^0 = \pi/2a$ is shifted to $k_F \neq k_F^0$ such that the vacuum is

shifted as a function of the magnetic field h_0. We introduce the new unitary transformation

$$V = e^{-(i/2)[(\delta-1)-q]\alpha_R} e^{-(i/2)[(\delta-1)+q]\alpha_L}. \tag{13}$$

As a result of the unitary transformation $\psi(x)$ is replaced by

$$V^\dagger \psi(x) V = e^{i(k_F^0+\pi q/L)x} R(x) + e^{-i(k_F^0+\pi q/L)x} L(x). \tag{14}$$

The value of q can be chosen such that the vacuum is shifted and the term $g\mu_B h_0 \sum(n(x) - 1/2)$ is canceled [14].

The effect of the unitary transformation given in Eq. (13) is to change the boundary condition for the eigenfunction in Eqs. (12),

$$V^\dagger |J\rangle |Q\rangle = |J)|Q), \tag{15a}$$

such that $\langle \alpha + 4\pi | J\rangle = \langle \alpha | J\rangle$ is replaced by

$$\langle \alpha + 4\pi | J) = e^{i(\delta-1)2\pi} \langle \alpha | J). \tag{15b}$$

Similarly we have for $\langle \beta | Q)$,

$$\langle \beta + 4\pi | Q) = e^{i(q-[q])2\pi} \langle \beta | Q), \tag{15c}$$

where $[q]$ is the closest integer to q.

Due to the boundary condition in Eq. (15b), the eigenvalues J are given by $J = n/2 + (\delta - 1)/2$, $n = 0, \pm 1, \pm 2, ...$ and $Q = m/2 + (q - [q])/2$, $m = 0, \pm 1, \pm 2, ...$ Using the unitary transformation given in Eq. (13) and the Bosonic representation in Eq. (10), we obtain the Bosonic form of the Hamiltonian:

$$\hat{H} = V^\dagger \tilde{H} V = V^\dagger \hat{U}^\dagger H \hat{U} V, \quad \hat{H} = \hat{H}_{n=0} + \hat{H}_{n\neq 0} + \Delta E_0, \tag{16a}$$

where $\hat{H}_{n=0}$ is the zero mode Hamiltonian

$$\hat{H}_{n=0} = \frac{\pi v}{L} (\hat{N}_R^2 + \hat{N}_L^2 + 2\gamma \hat{N}_R \hat{N}_L), \tag{16b}$$

where $v \equiv J_{\perp}a(1 + J_{\parallel}/\pi J_{\perp})$ and $\gamma = 2J_{\parallel}a/\pi v$. $\hat{H}_{n\neq 0}$ is the non-zero mode Bosonic Hamiltonian [14]

$$\hat{H}_{n\neq 0} = \int_0^L dx \left\{ \frac{\hat{v}}{2} \left[K(\partial_x \hat{\phi})^2 + \frac{1}{K}(\partial_x \hat{\theta})^2 \right] + \lambda \cos(2\alpha + \sqrt{16\pi}\hat{\theta} + \frac{4\pi}{L}\hat{Q}x + 4(K_F - K_F^0)x) \right\}, \tag{16c}$$

where $\hat{v} \equiv v\sqrt{(1-\gamma^2)}$, $K = \sqrt{(1-\gamma)/(1+\gamma)}$, $\lambda \equiv \hat{\lambda}/a^2$, $\hat{\lambda} \equiv J_{\parallel}a/2\pi^2$ with $\hat{\phi} = \hat{\theta}_L - \hat{\theta}_R$, $\hat{\theta} = \hat{\theta}_L + \hat{\theta}_R$. Here q is chosen such that the linear term $g\mu_B h_0$ is canceled and the ground state energy is shifted by,

$$\Delta E_0 = -\frac{\pi L}{2a}\frac{1}{J_{\perp}}\frac{g\mu_B h_0}{1 + 5J_{\parallel}/\pi J_{\perp}}, \quad 2\Delta K_F \equiv 2(K_F - K_F^0) \equiv 2\pi q/L = -\frac{2\pi}{L}\frac{g\mu_B h_0}{J_{\perp}(1 + 5J_{\parallel}/\pi J_{\perp})}. \tag{17}$$

Equation (17) shows that the shift in the Fermi momentum depends on the ratio between the Zeeman energy to the spin liquid lowest state excitation energy.

The spin current is obtained from the Hamiltonian given in Eq. (16). The Heisenberg equation of motion for the zero modes introduced in Ref. [13] allows us to construct the spin current operator. We replace the electric charge by the magnetic charge $s = g\mu_B$. As a result the spin current is given by the time derivative of the zero mode coordinate:

$$I_M = \frac{s}{2\pi}\frac{d\alpha}{dt} = \frac{s}{ih}[\alpha, \hat{H}] = \frac{sv_s}{L}(\hat{N}_L - \hat{N}_R), \quad v_s \equiv \frac{J_{\perp}a}{\hbar}\left(1 - \frac{J_{\parallel}}{\pi J_{\perp}}\right). \tag{18}$$

From Eq. (18) we see that the spin current is determined by the spin liquid velocity $v_s = J_{\perp}a/\hbar$ and the expectation value of the current operator $\hat{J} = \hat{N}_L - \hat{N}_R \equiv -i2d/d\alpha$.

To calculate the spin current we find the ground state wavefunction given by the eigenstate of the zero mode Hamiltonian. The zero mode Hamiltonian is obtained as a result of projecting out the non-zero mode field $\hat{\theta}(x)$. This projection is obtained with the help of the Renormalization Group (R.G.) equations given in Ref. [14]. We consider the mesoscopic region, namely that the thermal length $L_T = \hbar v_s/k_B T$ is larger than the circumference L of the ring. L defines the ring temperature $T^{ring} = \hbar v_s/k_B L$. The backward coupling is renormalized and replaced by λ_{eff}. At a scale b we have: $\hat{\lambda}(b) \sim \hat{\lambda}b^{-x}$ where $1 < b < L/a$ and $x = 2(2K-1)$. For $J_{\perp} > J_{\parallel}$, $x > 0$ and $\lambda(b)$ decreases. For $J_{\perp} \sim J_{\parallel}$ we have $x \simeq 0$ and $\hat{\lambda}(b) \simeq \hat{\lambda}/(1 + \hat{\lambda} ln(b))$ decreases much more slowly.

When $J_\parallel > J_\perp$, $\hat{\lambda}(b)$ grows with b at the scale $b^* \simeq e^{1/\hat{\lambda}(0)}$. When $\hat{\lambda}(b)$ diverges, it signals that a *spin gap* is opened. Thus for a finite ring, *gapless excitations* are possible in the regime $\pi J_\perp > J_\parallel > J_\perp$ if the length of the ring L is shorter than $\exp(1/\hat{\lambda})$, $L < a\exp(1/\hat{\lambda})$, and the thermal length L_T obeys $L_T > \exp(1/\hat{\lambda}) > L$.

Once the particle hole boson $\hat{\theta}(x)$ has been integrated away, one obtains a zero mode Hamiltonian. The zero mode eigenfunctions $\psi_{n,m}(\alpha,\beta)$ of the Hamiltonian given in Eq. (16) have the form:

$$\psi_{n,m}(\alpha,\beta) = Ae^{i(\delta-1)\alpha/2}e^{i(q-[q])\beta/2}e^{in\alpha/2}e^{im\beta/2}\left[1+\sum_{r=\pm2,\pm4,...} C_r e^{ir\alpha/2}\right]. \tag{19}$$

For simplicity we consider the case that q is an integer, $q-[q]=0$. Under this condition the wavefunction in Eq. (19) is characterized by $n = 0, \pm1, \pm2, ...$ and $m = 0$.

The effective zero mode Hamiltonian for $Q = 0$ has the form:

$$\hat{H}_{eff} = \frac{\hbar\pi v_s}{2L}\hat{J}^2 + \lambda_{eff}\cos(2\alpha + 2\pi q), \tag{20}$$

where $\lambda_{eff} = (J_\parallel/2\pi)F(L/a)[\sin(2\pi q)/(2\pi q)]$. Here $F(L/a)$ is the scaling function

$$F(L/a) = (L/a)^{-x}, \quad x > 0 \quad and \quad F(L/a) = (1+\hat{\lambda}ln(L/a))^{-1}, \quad x \sim 0.$$

Here, q measures the ratio of the Zeeman energy and the spin liquid lowest energy state excitation in the ring. When $q \ll 1$ ($h_0 \to 0$) the last term in Eq. (20) is replaced by $\cos 2\alpha$. For large magnetic fields, $q \simeq 0.5$ the cosine term can be ignored and the spin current is given by $I_M \simeq -g\mu_B(v_s/L)(1-\Omega(\theta_0)/\pi)$. Using typical values of exchange energy $J \sim 100K - 1000K$ and magnetic fields of $5-10$ Tesla one finds $q \sim 1$ justifying the neglect of the Ising term. In the remaining discussion we consider the modification of the current caused by the term $\lambda_{eff}\cos(2\alpha + 2\pi q)$.

Using Bloch theory we compute the eigenfunctions of the Hamiltonian in Eq. (20). We find $\psi_{n,l}(\alpha) = (1/\sqrt{4\pi})\exp(i(\delta-1)\alpha/2)\exp(in\alpha/2)U_{n,l}(\alpha)$; $n = 0,1,2,3$ are the Bloch states in the reduced zone and l is the energy band index. We restrict the discussion to the sector $Q = 0$. Therefore the possible states for J are $n = 0, 2$. Odd states are excluded since

they give states with $Q \neq 0$. Using Bloch states $\psi_{n,l}(\alpha) \equiv \langle \alpha | n, l)$ with eigenvalues $E_{n,l}$, we compute the spin current formula using Eq. (18) and find

$$I_M = -g\mu_B \frac{v_s}{L} \sum_{n=0}^{2} \sum_{l} \frac{1}{Z} (n, l|\hat{J}|n, l) e^{-E_{n,l}/k_B T} = -2g\mu_B \sum_{n=0}^{2} \sum_{l} \frac{1}{Z} \left(\frac{\partial E_{n,l}}{\partial n} \right) e^{-E_{n,l}/k_B T}. \quad (21)$$

Here Z is the zero mode partition function $Z = \sum_{n=0}^{2} \sum_l e^{-E_{n,l}/k_B T}$ and $|n, l)$ are Bloch states with the eigenvalues $E_{n,l}$. The second part of Eq. (21) was obtained from Bloch's theory, see Ref. [15]. Using Eq. (21) we calculate the current for two cases: (a) weak coupling, $\lambda_{eff}/(\hbar \pi v_s/2L) \ll 1$, and (b) strong coupling $\lambda_{eff}/(\hbar \pi v_s/2L) \gg 1$.

(a) The weak coupling case: Because λ_{eff} decreases as L^{-x}, $x < 1$ and the kinetic term (the term proportional to $\hat{J}^2$) decreases as $1/L$, the weak coupling region is achieved only for $J_\perp \gg J_\parallel$ or large magnetic fields $|\sin 2\pi q/2\pi q| \ll 1$. For this case we can use first order perturbation theory [16] to find the ground state Bloch wavefunction $\psi_{n=0}(\alpha)$ for the Hamiltonian in Eq. (20):

$$\psi_{n=0}(\alpha) = e^{i(\delta-1)\alpha/2} \left[1 + \frac{\lambda_{eff} L}{\pi v_s} \left(\frac{e^{-i2\alpha} e^{i2\pi q}}{2 - (\delta - 1)} - \frac{e^{i2\alpha} e^{-i2\pi q}}{(\delta - 1) + 2} \right) \right]. \quad (22)$$

Substituting Eq. (22) in the first part of Eq. (21) gives for the spin current

$$I_M = -g\mu_B \frac{v_s}{L} \left(1 - \frac{\Omega(\theta_0)}{\pi} \right) \left[1 - \frac{1}{(4\pi^3)^2} \left(\frac{J_\parallel/J_\perp}{1 - J_\parallel/\pi J_\perp} \right)^2 \left(\frac{L}{a} \right)^{2(1-x)} \left(\frac{\sin 2\pi q}{2\pi q} \right)^2 \right], \quad (23)$$

where $v_s = 2J_\perp a(1 - J_\parallel/\pi J_\perp)$ and $1 > x > 0$. Equation (23) is valid for values of $J_\parallel/J_\perp$ satisfying $(J_\parallel/J_\perp)|\sin 2\pi q/2\pi q| < (a/L)^{1-x}$.

(b) The strong coupling case: For this case we compute the eigenfunctions of Eq. (20) using the "instanton" method given in Ref. [17] or the solution given on page 231 in Ref. [15]. For large λ_{eff} the eigenvalues $E_{n,l=\pm}$ are given by $E_{n,\pm} = (\hbar\omega/2) \pm (\hbar\omega\sqrt{D}/\pi)\cos[2\pi n + 2\pi(\delta - 1)]$, where the energy scale $\hbar\omega$ is determined by $\hbar\omega \equiv (\hbar v_s^{(0)}/L)[(1/2\pi)(J_\parallel/J_\perp)(L/a)^{1-x}|\sin 2\pi q/2\pi q|]$ with $v_s^{(0)} = J_\perp a/\hbar$ and the band width $\sqrt{D}$ is given by $\sqrt{D} \equiv [\exp -\omega \int_0^{2\pi} \sqrt{2\cos(\alpha + 2\pi q)} d\alpha]$. Using the spin current given by Eq. (21) we find,

$$I_M = -g\mu_B \frac{v_s^{(0)}}{L} \sin[2\pi(1 - \Omega(\theta_0)/\pi]\sqrt{D}\sqrt{(8/\pi)(J_\parallel/J_\perp)|sin(2\pi q)/2\pi q|(L/a)^{1-x}}. \quad (24)$$

From Eq. (24) we see that when $J_{\parallel}/J_{\perp}$ increases the current is suppressed since the band width $\sqrt{D}$ scales like $\exp(-\sqrt{J_{\parallel}/J_{\perp}})$. The results in Eq. (23) and (24) are a consequence of the twist of the boundary condition obtained in Eq. (8c). In both cases the current was obtained for the even case, namely $\Delta_N = 0$. For the odd case $\Delta_N = 1$ and instead of $1 - \Omega(\theta_0)/\pi$ we have $\Omega(\theta_0)/\pi$.

Whether it is possible to detect the spin liquid current is an important question. Because of the Aharanov-Casher effect, when a metallic rod is inserted at the center of the ring one expects to find an electrostatic potential $\Phi_E(0) \simeq (I_M^{(0)}/L)(2\pi)^2$, i.e., a voltage proportional to the spin current $I_M^{(0)}$. For a magnetic field $h_0 \simeq 10$ Tesla, an exchange energy $J_{\perp} \sim 2000$ Kelvin and a ring of 10 μm, the Zeeman/exchange energy ratio is of the order of 10 allowing us to neglect the Ising term $J_{\parallel}$ and obtain a maximal current $I_M^{(0)}$. For a ring of 10 μm we need to have a low temperature such that $L_T = \hbar v_s/k_B T > 10$ μm; corresponding to a temperature of one Kelvin.

Materials that can be used to fabricate antiferromagnetic spin-1/2 ring structures include copper-benzoate. It is possible to fabricate such rings using plasma etching techniques in which a mesoscopic ring structure is etched out on a substrate and the desired magnetic material is deposited creating a magnetic ring structure. The persistent spin current can be measured using a scanning capacitance probe. Due to spin-orbit coupling the current will generate an electric field and therefore a potential difference. A small current in the ring will generate a voltage of the order of a nano-volt. An alternative approach is to attach two leads to a Luttinger liquid ring in the Aharonov-Bohm geometry with a crown shaped magnetic field. This generates a phase both in spin and charge. If the ring in the center is threaded with a flux tube [18] that exactly cancels the magnetic flux due to the crown shaped field, the remaining contribution is due to spin. Note that this flux tube will not affect the magnetic field on the ring.

In conclusion, we have shown that a crown shaped magnetic field when acting on a spin-1/2 Heisenberg antiferromagnetic chain can give rise to a spin current proportional to the solid angle $\Omega(\theta_0) = \pi(1 - \cos\theta_0)$. The amplitude of the current is controlled by the ratio

$J_\parallel/J_\perp$ and the Zeeman energy. A possible experimental realization has been suggested.

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REFERENCES

[1] M. Buttiker, Y. Imry, and R. Landauer, Phys. Lett. A **96**, 365 (1983).

[2] H.F. Cheung, Y. Gefen, and E.K. Riedelt, IBM J. Res. Dev. **32**, 359 (1988).

[3] L.P. Levy, G. Dolan, J. Dunsmuir, and H. Bouchiot, Phys. Rev. Lett. **64**, 2074 (1990).

[4] D. Loss, Phys. Rev. Lett. **69**, 343 (1992).

[5] D. Schmeltzer, Phys. Rev. B **47**, 7591 (1993).

[6] D. Schmeltzer and R. Berkovitz, Phys. Lett. A **253**, 341 (1999).

[7] D. Schmeltzer, Phys. Rev. B **63**, 125332 (2001).

[8] M.V. Berry, Proc. Royal Soc. (London) A **392**, 45 (1984).

[9] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).

[10] M. Nakahara, *Geometry, Topology and Physics*, pp. 364–372 (Institute of Physics Publ., Bristol, UK, 1990).

[11] F. Schütz, M. Koller, and P. Kopietz, Phys. Rev. Lett. **91**, 017205 (2003); cond-mat/0308230.

[12] E. Fradkin, *Field Theories of Condensed Matter Systems*, pp. 64–74 (Addison-Wesley, Redwood City, 1991).

[13] D. Schmeltzer, Phys. Rev. Lett. **85**, 4132 (2000).

[14] P. Sun and D. Schmeltzer, Phys. Rev. B **61**, 349 (2000).

[15] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics*, Part 2, Eq, (55.14), p. 226 (Pergamon Press, London, 1981).

[16] L.I. Schiff, *Quantum Mechanics*, 3rd edition (McGraw-Hill, Tokyo, 1968).

[17] R. Rajaraman, *Solitons and Instantons*, p. 316 (North Holland, Amsterdam, 1982).

[18] K. Hallberg, A. A. Aligia, A. P. Kampf, and B. Normand, unpublished.

FIGURES

FIG. 1. The geometry of a spin ring with circumference L in a crown shaped magnetic field $\vec{h}(x)$. The coordinate x is along the ring. The field has a tilt angle of θ_0 and subtends a solid angle $\Omega(\theta_0)$.